Algebraic geometry 1

Exercise Sheet 7

PD Dr. Maksim Zhykhovich Winter Semester 2025, 05.12.2025

Exercise 1. Let $X \subset \mathbb{A}^n$ be a quasi-affine algebraic set and $Y \subset \mathbb{P}^m$ a quasi-projective algebraic set. Let $\varphi : X \to Y$ be a map. Show that the following are equivalent.

(1) φ is a morphism (Definition 4.29 from the lecture)
(2) For every $p \in X$ there exists an open neighbourhood $U \subset X$ of p and polynomials $G_0, \ldots, G_m \in K[X_1, \ldots, X_n]$ with no common zero on U such that

$$\varphi(x) = [G_0(x), \ldots, G_m(x)] \text{ for all } x \in U .$$

Exercise 2. Denote by $\mathrm{Gr}(2,4)$ the set of 2-dimensional linear subspaces in K^4. We define a set-theoretic map $\varphi : \mathrm{Gr}(2,4) \to \mathbb{P}^5$ as follows. For every 2-dimensional subspace $V \in K^4$ choose two generators $v = (a_1, a_2, a_3, a_4)^t$, $w = (b_1, b_2, b_3, b_4)^t$ of V and set

$$\varphi(V) := [p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34}] \in \mathbb{P}^5 , \text{ where } p_{ij} = \det \begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} ,$$

that is p_{ij}, $1 \leq i < j \leq 4$, are exactly all maximal minors of the 4×2 matrix M with columns v and w.

(1) Show that φ is well-defined (that is independent of the choice of v and w).

(2) Denote by U_{12} the open subset $\{p_{12} \neq 0\}$ in $\mathbb{P}^5$ and set $W_{12} := \varphi^{-1}(U_{12})$. Show that the restricted map $\varphi : W_{12} \to U_{12}$ is injective and the image is closed in U_{12}.

Hint: Identify W_{12} with $\mathbb{A}^4$ by making the first 2×2 block of M equal to the identity. Identify U_{12} in a standard way with $\mathbb{A}^5$ and write down explicitely the map $\varphi : \mathbb{A}^4 \to \mathbb{A}^5$.

(3) Show that $\varphi : \mathrm{Gr}(2,4) \to \mathbb{P}^5$ is injective and the image is a degree 2 hypersurface (that is given by one homogeneous irreducible polynomial of degree 2). We identify $\mathrm{Gr}(2,4)$ with its image in $\mathbb{P}^5$.

(4) Let L be a 1-dimensional subspace in K^n (a line in $\mathbb{A}^4$ passing through the origin). Show that $X_L := \{$2-dimensional subspaces in K^n containing $L\}$ is a closed subset in $\mathrm{Gr}(2,4)$.

(5) Describe X_L for L generated by $(1,0,0,0)^t$ and for L generated by $(1,1,0,0)^t$.

(6) Show that $X_L \simeq \mathbb{P}^2$ for any L.